EXOTIC OPEN 4-MANIFOLDS WHICH ARE NON-LEAVES

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We study the possibility of realizing exotic smooth structures on punctured simply connected 4-manifolds as leaves of a codimension one foliation on a smooth compact manifold. In particular, we show the existence of an uncountable set of smooth open 4-manifolds which are not diffeomorphic to any leaf of a codimension one transversely C^2 foliation on a compact manifold. These examples include some exotic $\mathbb{R}^4$'s and exotic cylinders $S^3 \times \mathbb{R}$. See [8] for the complete paper.

Our results involve a set $\mathcal{Y}$ of smooth open 4-manifolds which we define below.

Theorem 1. *If $Y \in \mathcal{Y}$ is a leaf in a $C^{1,0}$ codimension one foliation of a closed 5-manifold, then it is a proper leaf and each connected component of the union of the leaves diffeomorphic to Y fibers over the circle with the leaves as fibers.*

Theorem 2. *For any manifold $Y \in \mathcal{Y}$ there exists an uncountable subset $\mathcal{Y}_Y \subset \mathcal{Y}$ of manifolds homeomorphic to Y that are not diffeomorphic to any leaf of a C^2 codimension one foliation of a compact manifold.*

The following result of independent interest, which uses the theory of levels and depth, will be used in the proof of Theorem 2.

Theorem 3. *The set of diffeomorphism classes of smooth manifolds of arbitrary dimension which are diffeomorphic to leaves of finite depth in C^2 codimension one foliations of compact manifolds is countable.*

Let us recall some important steps in the history of leaves and non-leaves. Cantwell and Conlon [3] showed that every open surface is diffeomorphic to a leaf of a foliation on every closed 3-manifold. The first examples of topological non-leaves were due to Ghys [5] and Inaba, Nishimori, Takamura, and Tsuchiya [7]. Later on, Attie and Hurder [1] constructed simply connected 6-dimensional non-leaves, among other results.

To define the set $\mathcal{Y}$ we need the concept of "end sum". Given two open smooth oriented 4-manifolds M and N with proper smooth embedded paths $c_1 : [0,\infty) \to M$ and $c_2 : [0,\infty) \to N$ defining ends of M and N, let V_1 and V_2 be tubular neighborhoods of $c_1([0,\infty))$ and $c_2([0,\infty))$. Then the end sum is $M \natural N = (M \setminus V_1) \bigcup_\partial (N \setminus V_2)$, where the boundaries, both diffeomorphic to $\mathbb{R}^3$, are identified so as to preserve the orientation. If N is homeomorphic to $\mathbb{R}^4$ then $M \natural N$ is homeomorphic to M. The end sum, up to diffeomorphism, depends only on the smooth proper isotopy classes of the curves. End sum was the first technique which made it possible to find infinitely many exotic structures on $\mathbb{R}^4$ [6] and it is an important tool for dealing with the problem of generating infinitely many smooth structures on open 4-manifolds [2, 4].

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An end of a smooth 4-manifold is *smoothly periodic* if there exists an unbounded domain $V \subset M$ homeomorphic to $S^3 \times (0,\infty)$ and a diffeomorphism $h : V \to V$ such that $h^n(V)$ defines the given end (i.e., $\{h^n(V)\}$ is a neighborhood base for the end).

It is well known that by removing a closed set carrying the 2-homology of the Kummer complex surface K3 it is possible to obtain a smooth 4-manifold $\mathbf{R}$ homeomorphic to $\mathbb{R}^4$ with an exotic end. For a homeomorphism $\psi : \mathbb{R}^4 \to \mathbf{R}$, we let $\mathbf{K}_t$ denote $\psi(D(t))$, where $D(t)$ is the standard closed 4-disk of radius t centered at the origin, so that its interior $\mathring{\mathbf{K}}_t$ has a smooth structure induced from $\mathbf{R}$ by ψ. Let $\natural\mathbf{R}_\infty = \natural_{i=1}^{\infty}\mathbf{R}$ be the infinite end sum. Then we have the following special case of Theorem 1.4 of Taubes [10].

Theorem (Taubes, [10]) *Let M be an open smooth simply connected 4-manifold with definite intersection form and exactly one end. If the end of M is homeomorphic to $S^3 \times (0,\infty)$ and smoothly periodic, then the intersection form is isomorphic to a diagonal form. As a consequence, for any homeomorphism $\psi : \mathbb{R}^4 \to \mathbf{R}$, there exists $r_0 > 0$ such that, for any $t, s > r_0$, $t \neq s$, $\mathring{\mathbf{K}}_t$ is not diffeomorphic to $\mathring{\mathbf{K}}_s$.*

Now we define $\mathcal{Y}$ to be the set of smooth manifolds Y (up to diffeomorphism) that are homeomorphic to simply connected compact 4-manifolds with finitely many punctures satisfying the following conditions:

(1) Y has an end diffeomorphic to the end of a non-trivial finite end sum $\natural_{i=1}^{k}\mathbf{R}$, to $\natural_{i=1}^{k}\mathring{\mathbf{K}}_t$ or to $\mathring{\mathbf{K}}_t\natural\mathbf{R}_\infty$ with $t > r_0$, and
(2) if $H_2(Y) = 0$, then Y has only one exotic end and the other ends (if there are any) are standard.
(3) In the particular case where Y is homeomorphic to $\mathbb{R}^4$ we only consider smooth structures with finite Taylor-index (See [11].)

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