Helicity is the only integral invariant of volume-preserving diffeomorphisms

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Let M be a compact 3-dimensional manifold without boundary, endowed with a Riemannian metric. We denote by $\mathfrak{X}^1_{\rm ex}$ the vector space of exact divergence-free vector fields on M of class C^1, endowed with its natural C^1 norm. We recall that a divergence-free vector field w is *exact* if the 2-form $i_w\mu$ is exact, where μ is the Riemannian volume form.

On exact fields, the curl operator has a well defined inverse $\operatorname{curl}^{-1} : \mathfrak{X}^1_{\rm ex} \to \mathfrak{X}^1_{\rm ex}$. The inverse of curl is a generalization to compact 3-manifolds of the Biot–Savart operator, and can also be written in terms of a (matrix-valued) integral kernel $k(x, y)$ as

$$\operatorname{curl}^{-1} w(x) = \int_M k(x,y)\, w(y)\, dy\,, \tag{1}$$

where dy stands for the Riemannian volume measure. Using this integral operator, one can define the helicity of a vector field w on M as

$$\mathcal{H}(w) := \int_M w \cdot \operatorname{curl}^{-1} w \; dx\,.$$

Here the dot denotes the scalar product of two vector fields defined by the Riemannian metric on M. It is well known that the helicity is invariant under volume-preserving diffeomorphisms, that is, $\mathcal{H}(w) = \mathcal{H}(\Phi_* w)$ for any diffeomorphism Φ of M that preserves volume (and orientation).

In view of the expression (1) for the inverse of the curl operator, it is clear that the helicity is an *integral invariant*, meaning that it is given by the integral of a density of the form

$$\mathcal{H}(w) = \int G(x, y, w(x), w(y))\, dx\, dy\,.$$

Our objective in this talk is to show, under some natural regularity assumptions, that the helicity is the only integral invariant under volume-preserving diffeomorphisms. To this end, let us define a regular integral invariant as follows:

Definition. *Let $\mathcal{I} : \mathfrak{X}^1_{\rm ex} \to \mathbb{R}$ be a C^1 functional. We say that $\mathcal{I}$ is a* regular integral invariant *if:*

1. *It is invariant under volume-preserving transformations, i.e.,* $\mathcal{I}(w) = \mathcal{I}(\Phi_* w)$ *for any diffeomorphism* Φ *of* M *that preserves volume (and orientation).*

2. *At any point* $w \in \mathfrak{X}^1_{\text{ex}}$*, the (Fréchet) derivative of* $\mathcal{I}$ *is an integral operator with continuous kernel, that is,*

$$(D\mathcal{I})_w(u) = \int_M K(w) \cdot u \,,$$

for any $u \in \mathfrak{X}^1_{\text{ex}}$*, where* $K : \mathfrak{X}^1_{\text{ex}} \to \mathfrak{X}^1_{\text{ex}}$ *is a continuous map.*

The following theorem shows that the helicity is essentially the only regular integral invariant in the above sense:

Theorem. *Let* $\mathcal{I}$ *be a regular integral invariant. Then* $\mathcal{I}$ *is a function of the helicity, i.e., there exists a* C^1 *function* $f : \mathbb{R} \to \mathbb{R}$ *such that* $\mathcal{I} = f(\mathcal{H})$*.*

The idea of the proof is that the invariance of the functional $\mathcal{I}$ under volume-preserving diffeomorphisms implies the existence of a continuous first integral for each exact divergence-free vector field. Because a generic vector field in $\mathfrak{X}^1_{\text{ex}}$ is not integrable, we conclude that the aforementioned first integral is a constant (that depends on the field), which in turn implies that $\mathcal{I}$ has the same value for all vector fields in a connected component of the level sets of the helicity. Because these level sets are path connected, the theorem follows.

References

[1] A. Enciso, D. Peralta-Salas, F. Torres de Lizaur, Helicity is the only integral invariant of volume-preserving transformations. Proc. Natl. Acad. Sci. 113 (2016) 2035–2040.