INDEPENDENT VARIATION OF SECONDARY CHARACTERISTIC CLASSES OF RIEMANNIAN FOLIATIONS

HIRAKU NOZAWA

Secondary characteristic classes of Riemannian foliations with framed normal bundle were introduced by Lazarov-Pasternack [LP76a]. It is a generalization of Chern-Simons invariants of framed Riemannian manifolds. Hurder [Hu81] showed that all variable classes of Lazarov-Pasternack vary independently based on a partial result due to Lazarov-Pasternack [LP76b]. Independent variation implies that the integral homology of the classifying space $FR\Gamma_q$ of codimension q Riemannian foliations with framed normal bundle surjects onto certain real vector space. Later Morita [Mo79] discovered new classes in terms of canonical Cartan connections, and showed that these new classes vary independently. The main result of this talk is the following.

Theorem 1. *All derivable classes of Lazarov-Pasternack and Morita are independently derivable.*

This generalizes results of Hurder and Morita mentioned above. We also show that $\pi_{q+1}(FR\Gamma_q)$ surjects to a real vector space and the universal homomorphism to $H^\bullet(FR\Gamma_q;\mathbb{R})$ is injective (Theorems 2 and 3 below are transversely Kähler analog of these results).

We prove the above results on Riemannian foliations by its transversely Kähler analog. Secondary characteristic classes of transversely Kähler foliations were introduced by Matsuoka-Morita [MM79]. Consider a differential graded algebra

$$KW_n = \bigwedge(u_1,\ldots,u_n)\otimes\left(\mathbb{R}[s_1,\ldots,s_n,\Phi]/\operatorname{Span}\{s_J\Phi^k \mid \deg J + k > n\}\right),$$

where

- $\deg\Phi = 2$, $\deg s_i = 2i$, $\deg u_i = 2i-1$, $d\Phi = 0$, $ds_i = 0$ and $du_i = s_i$.
- Φ corresponds to the basic Kähler class, s_i corresponds to the trace of the i-th power of the curvature of the normal bundle and u_i is the transgression of s_i.

For a manifold M with complex codimension n transversely Kähler foliation $\mathcal{F}$ with framed normal bundle, Matsuoka-Morita's construction yields a characteristic homomorphism

$$\Delta_{\mathcal{F}} : H^\bullet(KW_n) \longrightarrow H^\bullet(M;\mathbb{R}).$$

A class $\alpha \in H^\bullet(KW_n)$ is called *rigid* if for any manifold M with a one parameter family $\{\mathcal{F}_t\}$ of transversely Kähler foliations, the class $\Delta_{\mathcal{F}_t}(\alpha)$ is constant with respect to t. Otherwise, α is called *derivable*. For $k > 0$, it is easy to see that a class of the form $[u_I s_J \Phi^k]$ is derivable by the dilatation of the Kähler form. By Heitsch's formula, Morita-Matsuoka proved that $[u_I s_J]$ is rigid if $\min I + \deg J > n+1$. Then the space of secondary classes which are potentially derivable are spanned by the following

(1) $[u_I s_J \Phi^k]$ such that $\min I + \deg J + k = n+1$.
(2) $[u_I s_J \Phi^k]$ such that $\min I + \deg J \geq n+1$ and $k > 0$.

The independent variation of the classes of the form (2) is proved by computing the characteristic classes of the simple foliation on the unitary frame bundle over certain union of products of complex tori and projective spaces considered by Hurder [Hu09]. We show the independent variation of the classes of the form (1) by computing the characteristic classes of the pull back to the unitary normal frame bundle of linear deformations of the Hopf fibration $S^{2n+1} \to \mathbb{C}P^n$ by using the X-connections of Baum-Bott [BB72]. We show the independent variation of all derivable classes in KW_n by combining these two computations. Let $FK\Gamma_n$ be the the classifying space of complex codimension n transversely Kähler foliations with framed normal bundle. As a consequence of this computation, we have that

Theorem 2. *There exists a surjective homomorphism*

$$\pi_{2n+1}(FK\Gamma_n) \longrightarrow \mathbb{R}^{v(n)},$$

where $v(n)$ is the dimension of the vector space generated by derivable classes of KW_n of degree $2n+1$.

Theorem 3. *The canonical characteristic homomorphism $H^\bullet(KW_n) \longrightarrow H^\bullet(FK\Gamma_n;\mathbb{R})$ is injective.*

References

[BB72] P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Differential Geom. **7**, no. 3-4 (1972), 279–342.

[Hu81] S. Hurder, *On the secondary classes of foliations with trivial normal bundles*, Comment. Math. Helv. **56** (1981), no. 2, 307–326.

[Hu09] ———, *Characteristic classes for Riemannian foliations*, Differential geometry, pp. 11–35, World Sci. Publ., Hackensack, NJ, 2009

[LP76a] C. Lazarov and J. Pasternack, *Secondary characteristic classes for Riemannian foliations*, J. Differential Geom. **11** (1976), 365–385.

[LP76b] ———, *Residues and characteristic classes for Riemannian foliations*, J. Differential Geom. **11** (1976), 599–612.

[MM79] T. Matsuoka and S. Morita, *On characteristic classes of Kähler foliations*, Osaka J. Math. **16** (1979), no. 2, 539–550.

[Mo79] S. Morita, *On characteristic classes of Riemannian foliations*, Osaka J. Math. *16*, no. 1 (1979), 161–172.

Hiraku Nozawa, Department of Mathematical Sciences, Colleges of Science and Engineering, Ritsumeikan University, Nojihigashi 1-1-1, Kusatsu, Shiga, 525-8577, Japan

E-mail address: hnozawa@fc.ritsumei.ac.jp