Extended Abstract

Reeb components with complex leaves and their symmetries

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§0. Introduction and rough statement of results

We compute the automorphism group of a Reeb component with complex leaves, assuming that the leaves are of complex dimension 1 and the Reeb component is given by the Hopf construction. After the computation, if the situation allows us, we discuss further on the realization of such Reeb components in a Levi flat hypersurface in a complex surface and also on the extendability of automorphisms to those of ambient complex surfaces.

For higher dimensional ones, basically we can compute them to a large degree, once we know the automorphism group of the boundary leaf. *e.g.*, Horiuchi computed them for complex leaf dimension 2. (He completed the computation when the holonomy is infinitely tangent to the identity, while in other cases the description becomes quite complicated,but it is possible.) In this talk the boundary leaf is an elliptic curve, so that we know its automorpshim group well.

Here an *automorpshism* means a smooth, foliation preserving diffeomorphism which is holomorphic between leaves.

The result shows different features depending on the character of the holonomy of the boundary leaf. We assume that the holonomy diffeomorphism $\varphi \in \mathit{Diff}^{\infty}([0,\infty))$ of the boundary leaf (*i.e.*, a generator of the holonomy group) is expanding at $x = 0$. We have the following three cases.

Case (1) : The linear part of the holonomy is non-trivial.

Case (2) : The linear part is trivial but the infinite jet is non-trivial.

Case (3) : The holonomy is expanding but is flat to the identity.

Theorem 1 The automorphism group AutR of a Reeb component R with complex leaves is

Case (1) : a 3 dimensional or 5 dimensional solvable Lie group,

Case (2) : an ∞-dimensional solvable Lie group,

Case (3) : an ∞-dimensional solvable Lie group or slightly more complicated depending on the centralizer $Z_{\varphi} = \{\exp(tX) \mid t \in \mathbb{R}\}$ in $\mathit{Diff}^{\infty}([0,\infty))$ for some smooth vector field X or not.

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§1. Structure of the group of automorphisms

Let $H = H_\lambda = \mathbb{C}^*/\lambda^{\mathbb{Z}}$ ($\lambda \in \mathbb{C}^*$, $|\lambda| > 1$) denote the boundary elliptic curve of the Reeb component R. We easily see the following.

Proposition 2 $AutR$ admits the following decompositions into extensions.

$$0 \to Aut(R,H) \to AutR \to Aut_0H \to 0$$

and

$$0 \to \mathcal{K}_{\lambda,\varphi} \to Aut_0H \to Z_\varphi \to 0$$

where Aut_0H denotes the identity component $\cong H \cong T^2$.

Proposition 3 In Case (1), thanks to Sternberg's linearization, and in Case (2), thanks to Takens' normal form, the Szekeres vector field for φ is smooth, and the centralizer Z_φ coincides with $\{\exp(tX) \mid t \in \mathbb{R}\} \cong \mathbb{R}$.

In Case (3) in general we only have $\varphi^{\mathbb{Z}} \subset Z_\varphi \subset \{\exp(tX) \mid t \in \mathbb{R}\}$ where X is the Szekerez vector field which is guranteed only of C^1 at the origin.

In the first sequence $AutR \to Aut_0H$ is obtained by the restriction to the boundary and $Aut(R,H)$ is defined to be the kernel. Then $Aut_0H \to Z_\varphi \subset Diff^\infty([0,\infty))$ is looking at the action on the transverse space. The kernel $\mathcal{K}_{\lambda,\varphi}$ is a translation inside each leaf in the interior, where all leaves are biholomorphic to the cmplex plane.

As a consequence the problem of determining the automorphism group is reduced to the computation of $\mathcal{K}_{\lambda,\varphi}$.

§2. Schröder's equation on $[0,\infty)$

The computation of the kernel $\mathcal{K}_{\lambda,\varphi}$ is nothing but solving the following Schröder type functional equation on the half line $[0,\infty)$.

$$\text{(I)} \qquad \beta \circ \varphi = \lambda\beta \qquad \beta \in C^\infty([0,\infty);\mathbb{C})\,.$$

If we consider this equation on the open half line $(0,\infty)$, we easily see that the solution space $\mathcal{Z}_{\lambda,\varphi} \subset C^\infty((0,\infty);\mathbb{C})$ is isomorphic to $C^\infty(S^1;\mathbb{C})$.

Main Theorem
The space $\mathcal{K}_{\lambda,\varphi}$ of solutions to the equation (I) is as follows.

Case (1) : $\mathcal{K}_{\lambda,\varphi} = \{cx^p\,;\, c \in \mathbb{C}\} \cong \mathbb{C}$ if $\lambda = \mu^p$ and $p \in \mathbb{N}$ where $\mu = \varphi'(0)$, and otherwise $\mathcal{K}_{\lambda,\varphi} = 0$.

Case (2) and (3) : $\mathcal{K}_{\lambda,\varphi} \cong \mathcal{Z}_{\lambda,\varphi}$, namely any $\beta \in \mathcal{Z}_{\lambda,\varphi}$ extends to $[0,\infty)$ by $\beta(0) = 0$ as a smooth function and is flat at $x = 0$.

About the proof of Main Theorem:

For Case (I), by Sternberg's linearization [St], it is nothing but to look for weighted homogeneous functions and the resuts is well-known.

For Case (2) and (3) two ways are possible. One allows us to prove two cases together. This unified treatment relies on the center manifold theorem or C^r-section theorem (cf. [Sh]).

Case (2) is also proven by relying on Takens' normal form [Ta] and Fourier expansion/series. In this prove, after taking the normal form we compute the solution quite explicitely for natural ODE's related to the functional equation (I). A functional equation is decomposed into a family of ODE's and then the the solutions are brought together into those of (I) by Fourier series.

This method does not work for Case (3), to which we can give antoher proof, which is not applicable to Case (2) in turn. We take higher order derivatives of the equation (I) and estimate the results in somewhat smart way. This enables us to verify the convergence of any higher order derivatives of $\beta \in \mathcal{Z}_{\lambda,\varphi}$ to 0 when $x \to 0+0$.

§3. Applications and discussions

Pasting two Reeb components of Case (3) a Reeb foliation on S^3 is constructed. Of course this method is generalized to construct foliations with complex leaves on lens spaces. In thses cases the boundary is common to two Reeb components R_1 and R_2.

Theorem 4 In the above cases, the automorphism group of the resultant foliation on a lens space or S^3 is the fibre product of $AutR_1$ and $AutR_2$ over Aut_0H.

If we start from a codimension one foliation with complex leaves, we can perform a usual *turbulization* and get a new Reeb component R. For this modification, any of Case (1), (2), or (3) is possible for R.

Theorem 5 For a tubulization wiht the new Reeb component R of Case (2) or (3), any of manifold as the identity outside R, namely, the automorphism group of the resultant foliation includes $AutR$.

If we construct a *Hopf surface* W by $(\mathbb{C}^2 \setminus \{O\})/T^{\mathbb{Z}}$ where $T(z,w) = (\lambda \cdot z, \mu \cdot w)$ with $\lambda \in \mathbb{C}^*$ and $|\lambda|, \mu > 1$, the real hypersurface $M^3 = (\mathbb{C} \times \mathbb{R} \setminus \{O\})/T^{\mathbb{Z}}$ is *Levi-flat* and composed of two Reeb components $R_\pm$ with Levi-foliations.

Theorem 6 Any element of $AutR_+$ or of $AutR_-$ extends to the ambient Hopf surface W as a holomorphic automorphism.

Discussions

Theorem 1 and Theorem 6 shows that Reeb components of Case (1) exhibits a character simlar to compact complex manifolds.

Still it should be confirmed whether if the automorphism extends to the ambient surface in the case where the Reeb component appears as a part of a Levi-flat hypersurface which bounds a Stein surface.

On the other hand, some recent works relying on the Ueda theory suggests that Reeb components of Case (3) can not be realized in Levi-flat real hypersurfaces.

Our results mildly seduces us to imagine that the same might apply to Case (2).

Similar results on 5-dimensional Reeb components with complex 2-dimensional leaves are obtained by T. Horiuchi in [Ho]. There a complete computaion of the automorphism groups of all Hopf surfaces is done base on Kodaira's classification. Combined with a slight extension of Main Theorem, up to dimension 5 we can compute the automorphism groups of Reeb components.

A detailed exposition of this talk is found in [HM].

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