Dynamics of the geodesic and horocycle flows for laminations by hyperbolic surfaces

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Most of the results reported here is taken from a joint work [1] with Matilde Martinez and Alberto Verjovsky.

Throughout the talk, $(M, \mathcal{F})$ is to be a *minimal* lamination by hyperbolic surfaces on a compact metrizable space M. Let $\hat{M}$ be the leafwise unit tangent bundle of M: $\hat{M} = \cup_{x \in M} T^1_x L_x$, where L_x is the $\mathcal{F}$-leaf at x. Then $\hat{M}$ has a 3-dimensional lamination $T^1\mathcal{F}$ obatined by the decomposition $\hat{M} = \cup_{L \in \mathcal{F}} T^1 L$. Since $\mathcal{F}$ is minimal, $T^1\mathcal{F}$ is also minimal. In fact, $T^1\mathcal{F}$ is the orbit foliation of a right $PSL(2,\mathbb{R})$ action. Let D, U and B be the subgroups of $PSL(2,\mathbb{R})$ consisiting, respectively, of diagonal, unipotent upper triangular, and upper triangular matrices. The flow of D (U) is called leafwise geodesic (horocycle) flow. We discuss dynamical properties of these flows as well as the B-actions. Let $\Pi : \hat{M} \to M$ be the canonical projection, X a closed B-invariant subset of $\hat{M}$, and μ any ergodic harmonic measure of $\mathcal{F}$ (a probability measure on M). Denote $M_{X,1} = \{x \in M \mid \sharp(\Pi^{-1}(x) \cap X) = 1\}$ and $M_{X,>1} = M \setminus M_{X,1}$. By the ergodicity either $\mu(M_{X,1}) = 1$ or $\mu(M_{X,>1}) = 1$. By an argument using the leafwise Brownian motion, suggested by É. Ghys, we get:

Lemma 1. If $\mu(M_{X,>1}) = 1$, then $X = \hat{M}$

Corollary 2. (1) There is a unique minimal set for the B-action. (2) There is a dense orbit for the B-action.

Theorem 3. The following (1) and (2) are equivalent. (1) There is a closed B-invariant subset X for which $M_{X,1} = M$. (2) The lamination $\mathcal{F}$ is the orbit foliation of a continuous locally free B-action on M.

It is not the case that $\mu(M_{X.1}) = 1$ implies $M_{X,1} = M$. An example is constructed on a 4-manifold M, using the Thurston hyperbolization of the mapping torus of a pseudo-Anosov homeomorphism.

Theorem 4. The B-action is minimal under (1) or (2): (1) $\mathcal{F}$ admits a holonomy invariant transverse measure. (2) $(M, \mathcal{F})$ is the foliated Z bundle over a closed surface $\Gamma \setminus \mathbb{H}^2$ given by a minimal and indisrete homomorphism $\phi : \Gamma \to \mathrm{Homeo(Z)}$ for some compact metrizable space Z. Here indiscrete means that there are elements $g_n \in \Gamma \setminus \{e\}$ such that $\phi(g_n) \to \mathrm{id}_Z$.

Theorem 5([2]). The U-flow is minimal under (1) or (2): (1) $\mathcal{F}$ is a Riemannian foliation which admits a nonplanar leaf. (2) $\mathcal{F}$ is a codimension one foliation and the B-action on $\hat{M}$ is minimal.

Theorem 6. Assume M is a closed manifold. The D-flow (leafwise geodesic flow) is structurally stable in the sense that any $T^1\mathcal{F}$-leaf preserving leafwise C^1-perturbation is topologically equivalent to the D-flow by a $T^1\mathcal{F}$-leaf preserving homeomorphisms.

References

[1] M. Matilde, S. Matsumoto, A. Verjovsky, "Horocycle flows for laminations by hyperbolic Riemann surfaces and Hedlund's thoerem," J. Modern Dynamics, 10(2016) 113-134.

[2] S. Matsumoto, "Remarks on the horocycle flows for foliations by hyperbolic surfaces," To appear in Proc. A. M. S.