Group orders, dynamics and rigidity

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1 Introduction

Let G be a group. A *left-order* on G is a total order invariant under left multiplication, i.e. such that $a < b$ implies $ga < gb$ for all $a, b, g \in G$. It is well known that a countable group is left-orderable if and only if it embeds into the group of orientation-preserving homeomorphisms of $\mathbb{R}$, and each left-order on a group defines a canonical embedding up to conjugacy, called the *dynamical realization*. For this reason, left-orders appear in the study of dynamics and foliations.

A *circular order* on G is defined by a *cyclic orientation cocycle* $c : G^3 \to \{\pm 1, 0\}$ satisfying the following conditions:

i) (non degeneracy) $c(g_1, g_2, g_3) = 0$ if and only if $g_i = g_j$ for some $i \neq j$

ii) (cocylce condition) $c(g_2, g_3, g_4) - c(g_1, g_3, g_4) + c(g_1, g_2, g_4) - c(g_1, g_2, g_3) = 0$ for all $g_1, g_2, g_3, g_4 \in G$.

iii) (left invariance) $c(g_1, g_2, g_3) = c(hg_1, hg_2, hg_3)$ for all $g_i, h \in G$.

For countable groups, there is also a *dynamical realization* associating to each circular order a canonical conjugacy class of embedding $G \to \mathrm{Homeo}_+(S^1)$. This correspondence is the starting point for a rich relationship between the algebraic constraints on G imposed by orders, and the dynamical constraints on G–actions on S^1 or $\mathbb{R}$.

Spaces of orders and actions. For fixed G, we let $\mathrm{LO}(G)$ denote the set of all left-orders on G, and $\mathrm{CO}(G)$ the set of circular orders. These spaces have a natural topology; that on $\mathrm{CO}(G)$ comes from its identification with a subset of the infinite product $\{\pm 1, 0\}^{G \times G \times G}$. Left orders are a special case of circular orders (degenerate cocycles) so $\mathrm{LO}(G) \subset \mathrm{CO}(G)$ can be given the subspace topology. This agrees with the topology previously studied by Sikora [5] and others. For this reason we focus primarily on circular orders here, treating left orders as a special case.

Due to the relationship, via dynamical realization, between circular orders on a countable group G and actions of G on S^1, it is natural to ask about the relationship between the two spaces $\mathrm{CO}(G)$ and $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$, where $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$ is the space of actions of G on S^1, with the compact open topology. Similarly, there should be some relationship between $\mathrm{LO}(G)$ and $\mathrm{Hom}(G, \mathrm{Homeo}_+(\mathbb{R}))$. In particular, one hopes to use the topology of $\mathrm{CO}(G)$ to study that of $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$, and vice versa. Following Sikora [5], we have that $\mathrm{LO}(G)$ and $\mathrm{CO}(G)$ are both compact, totally disconnected and, for countable groups G, metrizable. Consequently, if $\mathrm{LO}(G)$ or $\mathrm{CO}(G)$ has no isolated points, then it is homeomorphic to a cantor set, and an important first question is thus to identify its isolated points. Even this can be highly nontrivial. We propose an approach by studying the realizations of isolated points in $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$.

2 Main results

Dynamical realization gives a map $\mathrm{CO}(G) \to \mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))/\sim$, where $\sim$ is the conjugacy relation. One can also define a partial inverse to this map. To what extent are these spaces related? As a first guess, one might (naively) propose the following.

Naive conjecture 2.1. Let G be a countable group. $\mathrm{CO}(G)$ has no isolated points if (or perhaps if and only if) $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$ is connected.

A supportive example is the case $G = \mathbb{Z}^2$. It is not difficult to show both that $\mathrm{Hom}(\mathbb{Z}^2, \mathrm{Homeo}_+(S^1))$ is connected and that $\mathrm{CO}(\mathbb{Z}^2)$ has no isolated points. In the case of the free group F_2 on two generators, $\mathrm{Hom}(F_2, \mathrm{Homeo}_+(S^1)) \cong \mathrm{Homeo}_+(S^1) \times \mathrm{Homeo}_+(S^1)$, which is also connected. This may have motivated the following conjecture stated in [1].

Conjecture 2.2. [1]. $\mathrm{CO}(F_2)$ has no isolated points.

It was also shown by Rivas in [4] that $\mathrm{LO}(F_2)$ has no isolated points, giving further evidence. However, we prove the following.

Theorem 2.3. [2]. $\mathrm{CO}(F_2)$ has has infinitely many isolated points. In fact, for any $n \geq 1$, the space $\mathrm{CO}(F_{2n})$ has infinitely many distinct classes of isolated points under the natural conjugation action of F_{2n} on $\mathrm{CO}(F_{2n})$.

This answers a question of [3] in the negative. The construction of the isolated orders in Theorem 2.3 is explicit and elementary, especially when described via their dynamical realizations – these are geometrically motivated "ping-pong" actions. The difficulty is in showing that these orders are indeed isolated points.

We remark that, since dynamical realizations are faithful, one might try to improve naive conjecture 2.1 by restricting to the subspace of *faithful* actions of a group on S^1. However, it is possible to show that the subset of faithful representations in $\mathrm{Hom}(F_2, \mathrm{Homeo}_+(S^1))$ is also connected. In fact, the relationship between $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$ and $\mathrm{CO}(G)$ is much more subtle. The aim of our work in [2] is to bring this relationship to light.

Dynamical characterization of isolated points. Our main theorem is a complete characterization of isolated points in $\mathrm{CO}(G)$ in terms of the dynamics of their dynamical realization.

Theorem 2.4. [2]. Let G be a countable group. A circular order on G is isolated if and only if its dynamical realization ρ is *rigid* in the following strong sense: for every action ρ' sufficiently close to ρ in $\mathrm{Hom}(G, \mathrm{Homeo}_+(S^1))$ there exists a continuous, degree 1 monotone map $h : S^1 \to S^1$ fixing the *basepoint* $x_0 \in S^1$ of the dynamical realization, and such that $h \circ \rho'(g) = \rho(g) \circ h$ for all $g \in G$.

There is an analogous statement for left orders and rigid actions on $\mathbb{R}$.

Theorem 2.4 is the main ingredient in the proof of Theorem 2.3, indeed, the isolated orders on free groups can be seen as an application. A major tool in the proof of Theorem 2.4 is an extension of work of Navas [3] on "maximal minimality" of dynamical realizations of left orders.

Further applications. We also give a detailed description of which faithful actions of a countable group G on S^1 can arise as dynamical realizations, leading to a new notion of the *linear part* of a circular order – a maximal, convex subgroup.

One can also move from isolated circular orders to isolated linear orders via central extensions by $\mathbb{Z}$. As a particular example, lifts of the rigid actions of F_2 on S^1 to actions on the real line give isolated left-orderings on the central extension $\mathbb{Z} \times F_2$, obtaining

Corollary 2.5. The pure braid group $P_3 \cong F_2 \times \mathbb{Z}$ has infinitely many distinct conjugacy classes of isolated left-orders.

3 Further questions

1. Give examples of other groups with isolated circular orders.

2. For $n > 1$ odd, do there exist (infinitely many?) isolated circular orders on F_n?

3. Are the examples given in the proof of Theorem 2.3 the only isolated circular orders on F_{2n}?

References

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