

NATURAL BOUNDARY VALUE PROBLEMS FOR WEIGHTED FORM LAPLACIANS

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Gradients in the sense of Stein and Weiss are $O(n)$ -irreducible parts of ∇ , the covariant derivative of an Riemannian manifold M of dimension n and of Riemannian metric g . For example, the bundle of differential p -forms is $O(n)$ -irreducible. But the target bundle of ∇ acting p -forms splits.

As a result, we obtain three $O(n)$ -gradients: d , δ and S . The first two are the familiar exterior derivative and coderivative. The third operator S completing the list, and defined just by the splitting, seems to be at least equally important. It is the only one of the three that has, like ∇ , an injective symbol. And that means the ellipticity. Roughly speaking, we can say that S is carrying the ellipticity of ∇ . S is called to be the *Ahlfors operator*.

In the particular case $p = 1$, the operator S , being the symmetric and trace free part of ∇ , is one of the most important operators in conformal geometry: conformal Killing forms, or - by duality- vector fields, constitute its kernel. It is worth to notice that, in the case of $M = \mathbb{R}^2 = \mathbb{C}$, the Ahlfors' operator becomes the Cauchy-Riemann one, so S may be treated as its higher (even odd) dimensional extension.

Recall that Ahlfors studied S as an operator acting on vector fields X in \mathbb{R}^n :

$$SX = \frac{1}{2}(DX + DX^t) - \frac{1}{n}\text{trace}(DX)I,$$

where $DX = (\partial X_i / \partial x_j)$, DX^t is the transpose of DX and $I = (\delta_{ij})$.

The adjoint operator is

$$(S^*\phi)_i = \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{ij}.$$

So, the resulting differential operator S^*S maps vector fields into vector fields.

In the case of an arbitrary Riemannian manifold it is more convenient to replace vector fields by their duals: one forms. S^*S may be then written in its invariant shape

$$S^*S = \frac{n-1}{n}d\delta + \frac{1}{2}\delta d - \text{Ric},$$

where Ric is the Ricci action on one-forms.

S^*S is strongly elliptic second order differential operator. In the case of Ricci flat manifold (and such is $M = \mathbb{R}^n$) it reduces to

$$L_{a,b} = ad\delta + b\delta d,$$

where a and b are positive constants. The operators of this form will be called the *weighted form Laplacians*. These operators give a subclass of the class of so called *non-minimal operators*. Notice that the last formula enables getting an extension of the action of $L_{a,b}$ onto skew-symmetric forms of any degree p .

The extended operator $L_{a,b}$ is just the subject of the talk. It would seem that $L_{a,b}$ theory were just a version of that one for the Laplace-Beltrami operator $\Delta = \delta d + d\delta = L_{1,1}$. But, when $a \neq b$, this is not the case. In contrast to the situation which pertains for Δ , the symbol of $L_{a,b}$, is no longer given by the metric tensor, so the situation is more subtle.

In the dimension three a version of $L_{a,b}$ acting on vector fields in a bounded domain was investigated in the context of an elastic body by H. Weyl. In particular, the boundary problem under three different, physically motivated, boundary conditions were solved there. Ahlfors solved the Dirichlet boundary problem for S^*S in the n -dimensional hyperbolic ball. He used there the fact that the group of Möbius transformations of the unit ball (i.e., the group of isometries with respect to the hyperbolic metric) acts transitively. A Poisson type centre formula he derived there enabled therefore getting the value of a solution at any point of the ball. In the case of the Euclidean ball there is no such a tool. Reimann solved the Dirichlet problem for $S^*S = 0$ on vector fields in this case. In analogy to the classical procedure for the Dirichlet problem for the Laplace operator, consisting in expanding the functions on the sphere into a series of spherical harmonics, he decomposed the space of vector fields into some suitably chosen $O(n)$ -invariant subspaces. Then he found a nice basis in each of the summands.

We are also going to adopt the Reimann's method here though his way of defining the Ahlfors operator of a higher order was passing to the space of trace-free symmetric tensors. Our way, in contrast to the Reimann one, is passing the space of skew-symmetric forms of an arbitrary degree p . The splitting onto $O(n)$ -invariant subspaces is then essentially different.

It seems to be interesting to find all the solutions for a complete list of some natural boundary conditions.

All conditions from the list are self-adjoint, and, in the case of elliptic gradients they constitute so called *elliptic boundary conditions* in the sense of Gilkey and Smith.

Let us describe shortly that rule. For any gradient G (one can think for a while that G is, e.g., d , δ or S but the formula is really very general) we have

$$(1) \quad (G^*G\omega_1, \omega_2) - (\omega_1, G^*G\omega_2) = \int_{\partial M} [g(\iota_\nu G\omega_1, \omega_2) - g(\omega_1, \iota_\nu G\omega_2)]$$

where $\iota_\nu G\omega$ is the contraction of $G\omega$ with the unit vector ν normal to the boundary ∂M . To make G^*G self-adjoint we have to accept boundary conditions annihilating the right hand side of (1), or stronger, annihilating each of the summands under the boundary integral. Now the unit normal ν will play its role. The original bundle we are dealing with (in our case the bundle of p -forms) is $O(n)$ -irreducible. But, it reduces at the boundary under the action of the subgroup $O(n-1)$ of $O(n)$ keeping ν invariant. As a result, by the Branching Rule, the original bundle splits at the boundary onto, say s , $O(n-1)$ -invariant subbundles. Denote by π^1, \dots, π^s the projections defined by the splitting. Then, by the orthogonality, $g(\omega_1, \iota_\nu G\omega_2)$ is equal to the sum

$$g(\pi^1\omega_1, \pi^1\iota_\nu G\omega_2) + \dots + g(\pi^s\omega_1, \pi^s\iota_\nu G\omega_2).$$

Now, there are 2^s candidates for elliptic boundary conditions, constructed as follows: For each $b = 1, \dots, s$, we choose *exactly* one of $\pi^b\omega_1$ and $\pi^b\iota_\nu G\omega_2$ and require it to vanish. For example, if we require to vanish the first multiplier in each summand we get the Dirichlet condition; if we require to vanish the other one we get the Neumann one. By other choices we get the whole their variety. The boundary conditions obtained that way seems to be in some sense “basic”, at least from the point of view of the representation theory. Of course, we realize that the list may not contain some other geometrically or physically important conditions like Robin one etc. Of course, for different purposes or for some physical applications, we may always perturb by lower order operators. When we do so, we need to worry about losing the symmetry condition for the boundary integrand in (1), i.e., about loosing the self-adjointness. These perturbations will possibly take the form of order 0 operators, added either to the interior operator G^*G , or to the boundary operator $\omega \mapsto \iota_\nu G\omega$.

According to Branson and Pierzchalski, there are four such conditions on the list in the case of $O(n)$ -gradients acting on the space of differential forms

of any degree on a Riemannian manifold M with an nonempty boundary ∂M :

Dirichlet boundary condition (\mathcal{D}):

$$\omega^T = 0 \quad \text{and} \quad \omega^N = 0 \quad \text{on} \quad \partial M.$$

Absolute Boundary condition (\mathcal{A}):

$$\omega^N = 0 \quad \text{and} \quad (d\omega)^N = 0 \quad \text{on} \quad \partial M.$$

Relative boundary condition (\mathcal{R}):

$$(\delta\omega)^T = 0 \quad \text{and} \quad \omega^T = 0 \quad \text{on} \quad \partial M.$$

The fourth boundary condition (\mathcal{B}):

$$(\delta\omega)^T = 0 \quad \text{and} \quad (d\omega)^N = 0 \quad \text{on} \quad \partial M.$$

Here ω^T and ω^N denote the tangent and the normal parts of ω at the boundary, respectively. The first three conditions are known to geometers. The fourth one seems to be unknown. But, being natural, it should have a geometric or physical meaning.

Observe also a surprising symmetry with respect to the Hodge star operator \star . Namely, by the following known relations:

$$\star\star = \pm 1, \quad (\star\omega)^T = \pm \star(\omega^N), \quad (\star\omega)^N = \pm \star(\omega^T)$$

and

$$\delta\omega = \pm \star d \star \omega, \quad d\omega = \pm \star \delta \star \omega$$

it follows easily that the set of all the four boundary conditions $\{\mathcal{D}, \mathcal{A}, \mathcal{R}, \mathcal{B}\}$ is star-invariant. More precisely, each of the conditions \mathcal{D}, \mathcal{B} is star-invariant, while the conditions \mathcal{A} and \mathcal{R} are star-symmetric each to the other.

We are going to solve all the four boundary problems $\mathcal{D}, \mathcal{A}, \mathcal{R}, \mathcal{B}$ for the operators $L_{a,b}$ acting onto differential forms of arbitrary degree p in the Euclidean unit ball in \mathbb{R}^n .

The talk is based on joint paper with Antoni Pierzchalski.