PRODUCING COMPACT INVARIANT SETS IN REEB FLOWS

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This is a joint work with T. Arai and Y. Kano.
Let (M, ξ) be a contact manifold. Then for each contact form α with $\operatorname{Ker}\alpha = \xi$ one can associate a unique flow, say ψ^α, on M called the Reeb flow of α. I am interested in the following question: Given (M, ξ), to what extent can we vary the dynamics of the Reeb flow by a change of α? In this talk, I exclusively consider $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$, where $\xi_{\text{std}} = \operatorname{Ker}\alpha_{\text{std}}$ and $\alpha_{\text{std}} = dz + \frac{1}{2}\sum_{j=1}^{n} {r_j}^2 d\theta_j$. Remark that the Reeb flow of α_{std} is generated by $\partial/\partial z$. I want to modify it so that it contains a compact invariant set of various type. Recently, Geiges-Röttgen-Zehmisch [2014] have realized an n-dimensional torus ($n \geq 2$) with an irrational linear flow as an invariant set of a Reeb flow of ξ_{std}. We generalize their result as follows.

Theorem. *Let φ be any flow on T^n which is obtained by a suspension of a diffeomorphism of T^{n-1}, and let $\mathcal{A} \subset T^n$ be any compact invariant set of φ. Then, we can find an embedding $T^n \subset \mathbb{R}^{2n+1}$ and a contact form α on $\mathbb{R}^{2n+1}$ with $\operatorname{Ker}\alpha = \xi_{\text{std}}$ such that :*

(1) *$\alpha = \alpha_{\text{std}}$ outside a small neighborhood of $\mathcal{A}$.*
(2) *The Reeb flow ψ^α restricted to $\mathcal{A}$ is orbit equivalent to φ.*
(3) *All orbits of ψ^α outside $\mathcal{A}$ are unbounded.*

For instance, when $n \geq 2$, one can take as $\mathcal{A}$ a transversely Cantor minimal set etc.
Some subsets of $\mathbb{R}^{2n+1}$ other than subsets of T^n are also realizable:

Proposition. *The generalized Hopf flows on S^{2n-1} are realizable in the sense that S^{2n-1} (instead of $\mathcal{A}$) satisfies the three properties in the above theorem. $S^{2k_1-1} \times \cdots \times S^{2k_p-1}$ ($k_1 + \cdots + k_p = n$) and $S^{2n-1} \times I^2$ are also realizable.*

Problem. What flows on what manifolds can be realized as a compact invariant sets of a Reeb flow in $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$? Find many more examples, or, develop a powerful method of realization.

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