THE EULER CLASS OF AN UMBILIC FOLIATION

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The main idea of this manuscript is to compute the Euler class of a foliation $\mathcal{F}$, assuming it admits a compact and umbilic leaf. Besides the umbilicity of the leaf, the geometrical assumptions considered are the sectional curvatures of the ambient manifold restricted to the leaves of $\mathcal{F}$, and they are the key to write explicitly this class. Translating geometrical hypothesis into topological ones implies obstructions to the existence of these foliations by looking at the cohomology of the ambient manifold as well as by asking for positiveness of sectional curvatures of M along $\mathcal{F}$.

Theorem A. *Let $\mathcal{D}^{2k}$ be a distribution on a Riemannian manifold M^{2k+p} with pure curvature form. Let L be a compact umbilic submanifold of M, with dimension $2k$, and suppose the sectional curvatures of M are nonnegative along L. If $\mathcal{D}$ is tangent to L, then $\epsilon(\mathcal{D}) \neq 0$.*

In order to remove the "pure curvature form" hypothesis, we consider Milnor's proof of Hopf conjecture on dimension four,

Theorem [Milnor]. *Let M be a compact orientable Riemannian manifold of dimension 4. If its sectional curvatures always have the same sign, $\chi(M) \geq O$. If the sectional curvature is always positive or always negative, $\chi(M) > 0$.*

Theorem B. *Let $\mathcal{D}^4$ be a distribution on a Riemannian manifold M^{4+p}. Let L be a compact umbilic submanifold of M, with dimension 4, and suppose the sectional curvatures of M are positive along L. If $\mathcal{D}^4$ is tangent to L, then $\epsilon(\mathcal{D}) \neq 0$.*

On the other hand Alain Connes introduced the Euler characteristic $\chi(\mathcal{F}, \nu)$ for a foliation endowed with a transverse measure. The particular case where $\mathcal{F}$ is determined by a closed and global form ν of its normal distribution, which is called SL-foliation, $\chi(\mathcal{F}, \nu)$ is shown to be nonnegative, provided the sectional curvatures of the leaves of $\mathcal{F}$ always have the same sign. It reads

Theorem C. *Let $\mathcal{F}$ be a SL-foliation of dimension 4 on a closed Riemannian manifold M^{4+p}. If the sectional curvatures of the leaves always have the same sign, then $\chi(\mathcal{F}, \nu) = \int_M \epsilon(\mathcal{F}) \wedge \nu \geq 0$.*

Applications of characteristic classes to foliations date back to theorems of J. Milnor and J. Wood, dealing with the Euler class as an obstruction to the existence of foliations transverse to the fibers of overly twisted circle bundles over surfaces. Regarding obstructions to integrability, a theorem of R. Bott asserts that given a codimention p distribution on the tangent bundle, a necessary condition for its integrability is the vanishing of all Pontryagin classes (associated to the normal bundle) of degrees higher than $2p$. In addition, J. Pasternack lowered the condition to degrees above p, assuming that the distribution is tangent to a Riemannian foliation.

Foliations are integrable subbundles of the tangent bundle, and although in the literature characteristic classes are constructed on the normal bundle, there are interesting consequences when they are computed on the tangent distributions themselves. In this context, geometrical and topological hypothesis on the foliations and on the ambient manifold are assumed in order to explicitly determine properties of the classes.

2010 *Mathematics Subject Classification.* 57R30, 57R20.
This work is part of I. Gonçalves' thesis. Supported by a scholarship from CNPq, number 141113/2013-8.

For example, if the foliation is totally geodesic and of odd dimension n, a theorem of A. Naveira asserts the $(n+p)$-th Pontryagin class of $\mathcal{F}$ vanishes. If the leaves are surfaces, and the normal distribution is a minimal foliation, then from F. Brito, the Euler class of $\mathcal{F}$ is different from zero when $\mathrm{Ric}(M) > 0$.

Umbilic foliations were studied from the perspective of conformal geometry by R. Langevin and P. Walczak. Their approach includes properties of local and global invariants, the question whether a Riemannian manifold admits an umbilic or a foliation with weaker conditions, such as Dupin foliations, as well as asking how far from umbilic a foliation is by defining a conformal invariant quantity. In dimension 3, they were classified in the light of transversely holomorphic fields by M. Brunella and E. Ghys.

Euclidean spheres do not admit totally geodesic nor umbilic foliations of codimension one. However, for codimention greater than one, those are far from being geometrically classified. The geometrical abundance is made explicit already in the codimension 2 case of S^3,

Theorem [Gluck-Warner]. *A submanifold of* $\widetilde{G}_2(\mathbb{R}) \cong S^2 \times S^2$ *corresponds to a fibration of* S^3 *by oriented great circles if and only if it is the graph of a certain distance decreasing map* $f : S^2 \to S^2$.

Umbilic foliations of S^3 and other odd spheres S^{2k+1} are obtained by taking a smooth positive function f constant on the leaves of a totally geodesic foliation and making a conformal change of the induced metric, $\langle \cdot, \cdot \rangle \mapsto f \langle \cdot, \cdot \rangle$, or by considering small deformations of all planes which give great circle fibrations, in order to obtain affine nonlinear planes intersecting the sphere.

References

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